

On the interpolation of discontinuous functions

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Abstract

Given a sequence of real numbers, we consider its subsequences converging to possibly different limits and associate to each of them an index of convergence which depends on the density of the associated subsequences. This index turns out to be useful for a complete description of some phenomena in interpolation theory at points of discontinuity of the first kind. In particular we give some applications to Lagrange and Shepard operators.

1 An index of convergence

The aim of this paper is to investigate the behavior of non converging sequences, for which we can find suitable converging subsequences. The density of the subsequences converging to a given limit determines an index of convergence in the sense of Definition 1.1. Our main aim is to use this index in order to obtain a complete description of the behavior of some sequences of interpolating operators on functions with a finite number of discontinuity of the first kind. This problem has been considered for a long time both for algebraic and trigonometric polynomials. While for trigonometric polynomials we have some classical completely satisfactory results, in the case of algebraic polynomials the situation is quite different. Some properties of Shepard operators on functions with a discontinuity of the first kind have been established in [1] and subsequent papers in terms of lower and upper limits, but the problem of a complete description remains substantially opened.

The introduction of the index of convergence in Definition 1.1 allows us to give a solution to this problem. One of the main properties of this index resides in the fact that a sequence may converge to different real numbers having indices in the interval $[0, 1]$. In the case where only one real number has index

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1, the concept can be related to that of statistical convergence considered in [5] and subsequently generalized in different ways (see, e.g., [6, 7, 2, 4]). Due to the particular formulation of the concept of statistical convergence and its subsequent extensions and generalizations, it has not been possible to use it for a deeper analysis of the interpolation of discontinuous functions.

In this section we define the index of convergence and give some of its properties and characterizations. In Sections 2 and 3 we consider the indices of convergence of Lagrange and respectively Shepard operators applied to functions having a finite number of points of discontinuity of the first kind.

Let $K \subset \mathbb{N}$; the lower density and, respectively, the upper density of K are defined by

$$\delta_-(K) := \liminf_{n \rightarrow +\infty} \frac{|K \cap \{1, \dots, n\}|}{n}, \quad \delta_+(K) := \limsup_{n \rightarrow +\infty} \frac{|K \cap \{1, \dots, n\}|}{n}.$$

In the case where $\delta_-(K) = \delta_+(K)$ the density of K is defined as follows

$$\delta(K) := \delta_-(K) = \delta_+(K).$$

We observe that $\delta_-(K) = 1 - \delta_+(K^c)$. Indeed

$$\begin{aligned} \delta_-(K) &= \liminf_{n \rightarrow +\infty} \frac{|K \cap \{1, \dots, n\}|}{n} = \liminf_{n \rightarrow +\infty} \frac{|K \cap \{1, \dots, n\}| + n - n}{n} \\ &= \liminf_{n \rightarrow +\infty} \left(1 - \frac{n - |K \cap \{1, \dots, n\}|}{n} \right) = 1 + \liminf_{n \rightarrow +\infty} \left(-\frac{|K^c \cap \{1, \dots, n\}|}{n} \right) \\ &= 1 - \limsup_{n \rightarrow +\infty} \left(\frac{|K^c \cap \{1, \dots, n\}|}{n} \right) = 1 - \delta_+(K^c). \end{aligned}$$

Similarly, it can be shown that $\delta_+(K) = 1 - \delta_-(K^c)$.

We are now in a position to make the following definition.

Definition 1.1 *Let $(x_n)_{n \geq 1}$ be a sequence of real numbers. For every real number L , the index of convergence of the sequence $(x_n)_{n \geq 1}$ to L is defined by*

$$i(x_n; L) := 1 - \sup_{\varepsilon > 0} \delta_+(\{n \in \mathbb{N} \mid x_n \in]-\infty, L - \varepsilon] \cup [L + \varepsilon, +\infty[\}).$$

Moreover, we also set

$$\begin{aligned} i(x_n; +\infty) &:= 1 - \sup_{M \in \mathbb{R}} \delta_+(\{n \in \mathbb{N} \mid x_n \in]-\infty, M]\}), \\ i(x_n; -\infty) &:= 1 - \sup_{M \in \mathbb{R}} \delta_+(\{n \in \mathbb{N} \mid x_n \in [M, +\infty[\}). \end{aligned}$$

Remark 1.2 We point out the following explicit expression of the index of

convergence of a sequence $(x_n)_{n \geq 1}$

$$\begin{aligned}
i(x_n; L) &= 1 - \sup_{\varepsilon > 0} \delta_+ (\{n \in \mathbb{N} \mid x_n \in] - \infty, L - \varepsilon] \cup [L + \varepsilon, +\infty[\}) = \\
&= 1 + \inf_{\varepsilon > 0} (-\delta_+ (\{n \in \mathbb{N} \mid x_n \in] - \infty, L - \varepsilon] \cup [L + \varepsilon, +\infty[\})) = \\
&= \inf_{\varepsilon > 0} (1 - \delta_+ (\{n \in \mathbb{N} \mid x_n \in] - \infty, L - \varepsilon] \cup [L + \varepsilon, +\infty[\})) = \\
&= \inf_{\varepsilon > 0} \delta_- (\{n \in \mathbb{N} \mid x_n \in]L - \varepsilon, L + \varepsilon[\}).
\end{aligned}$$

Definition 1.1 can be extended as follows.

We set for brevity $B_\varepsilon :=] - \varepsilon, \varepsilon[$ whenever $\varepsilon > 0$.

Definition 1.3 Let $(x_n)_{n \geq 1}$ be a sequence of real numbers and let A be a subset of \mathbb{R} . We define index of convergence of $(x_n)_{n \geq 1}$ relatively to A

$$i(x_n, A) := 1 - \sup_{\varepsilon > 0} \delta_+ (\{n \in \mathbb{N} \mid x_n \notin A + B_\varepsilon\}).$$

Also in this case we have the following expression of the index of convergence

$$i(x_n, A) = \inf_{\varepsilon > 0} \delta_- (\{n \in \mathbb{N} \mid x_n \in A + B_\varepsilon\}).$$

Example 1.4 i) As a simple example, we can take $x_n := \cos n\pi/2$, it is easy to recognize that

$$i(x_n; 0) = \frac{1}{2}, \quad i(x_n; 1) = \frac{1}{4}, \quad i(x_n; -1) = \frac{1}{4}.$$

ii) As a further example, let $\alpha \in [0, 1[$ be irrational, $\beta \in [0, 1[$ and consider

$$x_n := n\alpha + \beta - [n\alpha + \beta] \quad (= n\alpha + \beta \mod 1)$$

where $[x]$ denotes the integer part of x .

The well-known equidistribution theorem of Weyl ensures that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(k\alpha + \beta \mod 1) = \int_0^1 f(t) dt$$

for each Riemann integrable function in $[0, 1]$. Then it follows that

$$\delta (\{n \in \mathbb{N} \mid x_n \in A\}) = |A|$$

for every Peano-Jordan measurable set $A \subset [0, 1[$, where $|\cdot|$ denotes the Peano-Jordan measure. Then

$$i(x_n; A) = |A|.$$

Remark 1.5 If the index of convergence of a sequence $(x_n)_{n \geq 1}$ to a real number L is equal to 1, we have

$$\sup_{\varepsilon > 0} \delta_+ (\{n \in \mathbb{N} \mid x_n \in]-\infty, L - \varepsilon] \cup [L + \varepsilon, +\infty[\}) = 0.$$

Hence, for all $\varepsilon > 0$

$$\begin{aligned} 0 &= \delta_+ (\{n \in \mathbb{N} \mid x_n \in]-\infty, L - \varepsilon] \cup [L + \varepsilon, +\infty[\}) \\ &\geq \delta_- (\{n \in \mathbb{N} \mid x_n \in]-\infty, L - \varepsilon] \cup [L + \varepsilon, +\infty[\}) \geq 0, \end{aligned}$$

consequently $\delta (\{n \in \mathbb{N} \mid x_n \in]-\infty, L - \varepsilon] \cup [L + \varepsilon, +\infty[\}) = 0$, which means that $(x_n)_{n \geq 1}$ converges statistically to L .

In the next proposition we point out some relations between the index of convergence to a number L and the density of suitable subsequences converging to L .

Proposition 1.6 *Let $(x_n)_{n \geq 1}$ be a sequence of real numbers and $\sigma \in]0, 1]$. Then $i(x_n, L) \geq \sigma$ if and only if there exists a subsequence $(x_{k(n)})_{n \geq 1}$ converging to L such that*

$$\delta_- (\{k(n) \mid n \in \mathbb{N}\}) \geq \sigma.$$

PROOF. \Rightarrow) For every $n \geq 1$, we consider the set $M_{1/n} := \{m \in \mathbb{N} \mid |x_m - L| < 1/n\}$. From Remark 1.2, for every $n \in \mathbb{N}$ there exists $\tilde{\nu}_n$ such that

$$\frac{|M_{1/n} \cap \{1, 2, \dots, j\}|}{j} \geq \sigma - \frac{1}{n}$$

whenever $j > \tilde{\nu}_n$. At this point we define recursively a new sequence $(\nu_n)_{n \geq 1}$ by setting $\nu_1 = \tilde{\nu}_1$ and $\nu_n = \max\{\tilde{\nu}_n, \nu_{n-1} + 1\}$. We have

$$\frac{|M_{1/n} \cap \{1, 2, \dots, j\}|}{j} \geq \sigma - \frac{1}{n} \text{ for all } j > \nu_n. \quad (1.1)$$

Consider the set of integers

$$K = \bigcup_{n \geq 1} (M_{1/n} \cap \{1, 2, \dots, \nu_{n+1}\})$$

and the subsequence $(x_n)_{n \in K}$.

For every $\varepsilon > 0$, let $m \in \mathbb{N}$ such that $1/m \leq \varepsilon$. Then for every $k \in K$ satisfying $k > \nu_m$ we have $k \in \bigcup_{n \geq m} (M_{1/n} \cap \{1, 2, \dots, \nu_{n+1}\})$ and hence $|x_k - L| < \frac{1}{m} \leq \varepsilon$. This shows that the subsequence $(x_n)_{n \in K}$ converges to L .

On the other hand, for every $j > \nu_m$, there exists $l \geq m$ such that $\nu_l < j \leq \nu_{l+1}$ and thanks to (1.1) we have

$$\begin{aligned} \frac{|K \cap \{1, 2, \dots, j\}|}{j} &\geq \frac{|M_{1/l} \cap \{1, 2, \dots, \nu_{l+1}\} \cap \{1, 2, \dots, j\}|}{j} \\ &= \frac{|M_{1/l} \cap \{1, 2, \dots, j\}|}{j} \geq \sigma - \frac{1}{l} \geq \sigma - \frac{1}{m} \geq \sigma - \varepsilon \end{aligned}$$

that is

$$\liminf_{n \rightarrow \infty} \frac{|K \cap \{1, 2, \dots, j\}|}{j} \geq \sigma.$$

\Leftarrow) We suppose that there exists a subsequence $(x_{k(n)})_{n \geq 1}$ converging to L such that $\delta_-(\{k(n) \mid n \in \mathbb{N}\}) \geq \sigma$. For every $\varepsilon > 0$ there exists $\nu_\varepsilon \in \mathbb{N}$ such that $|x_{k(n)} - L| < \varepsilon$ whenever $n \geq \nu_\varepsilon$. Hence

$$\begin{aligned} \delta_-(\{n \in \mathbb{N} \mid |x_n - L| < \varepsilon\}) &\geq \delta_-(\{n \in \mathbb{N} \mid |x_{k(n)} - L| < \varepsilon\}) \\ &= \delta_-(\{k(n) \mid n \geq \nu_\varepsilon\}) \\ &= \delta_-(\{k(n) \mid n \in \mathbb{N}\}) \geq \sigma \end{aligned}$$

and therefore, from Remark 1.2, we obtain $i(x_n, L) \geq \sigma$. \square

Proposition 1.7 *Let $(x_n)_{n \geq 1}$ be a sequence of real numbers and $(A_m)_{m \geq 1}$ a sequence of subsets of \mathbb{R} such that $\overline{A_k} \cap \overline{A_j} = \emptyset$ for all $k \neq j$. Then*

$$0 \leq \sum_{k=1}^{+\infty} i(x_n, A_k) \leq 1.$$

In particular, if $(L_m)_{m \geq 1}$ is a sequence of distinct elements of $[-\infty, \infty]$ such that, for every $m \geq 1$

$$i(x_n; L_m) = \alpha_m,$$

for some $\alpha_m \geq 0$, then

$$0 \leq \sum_{k=1}^{+\infty} \alpha_k \leq 1.$$

PROOF. Let $N \geq 1$; since $\overline{A_k} \cap \overline{A_j} = \emptyset$ whenever $k \neq j$, we can choose ε such that

$$(A_k + B_\varepsilon) \cap (A_j + B_\varepsilon) = \emptyset$$

for all $k, j = 1, \dots, N$, $k \neq j$.

Now consider the set

$$M_\varepsilon^{(k)} := \{n \in \mathbb{N} \mid x_n \in A_k + B_\varepsilon\}$$

and observe that $M_\varepsilon^{(k)} \cap M_\varepsilon^{(j)} = \emptyset$ whenever $k, j = 1, \dots, N$, $k \neq j$. Then we can conclude that

$$\begin{aligned} 0 &\leq \sum_{k=1}^N i(x_n, A_k) \leq \sum_{k=1}^N \delta_-(\{n \in \mathbb{N} \mid x_n \in A_k + B_\varepsilon\}) \\ &= \sum_{k=1}^N \liminf_{n \rightarrow \infty} \frac{|M_\varepsilon^{(k)} \cap \{1, \dots, n\}|}{n} \leq \liminf_{n \rightarrow \infty} \left(\sum_{k=1}^N \frac{|M_\varepsilon^{(k)} \cap \{1, \dots, n\}|}{n} \right) \\ &= \liminf_{n \rightarrow \infty} \frac{\left| \bigcup_{k=1}^N M_\varepsilon^{(k)} \cap \{1, \dots, n\} \right|}{n} = \delta_-\left(\bigcup_{k=1}^N M_\varepsilon^{(k)} \right) \leq 1 \end{aligned}$$

\square

Remark 1.8 Observe that if in the preceding Proposition we have $\sum_{k=1}^{+\infty} \alpha_k = 1$, then every subsequence $(x_{k(n)})_{n \geq 1}$ of $(x_n)_{n \geq 1}$ which converges to a limit L different from L_m , $m \geq 1$, necessarily satisfies $\delta_-(\{k(n) \mid n \in \mathbb{N}\}) = 0$ and therefore $i(x_n; L) = 0$.

Indeed, if a subsequence $(x_{k(n)})_{n \geq 1}$ of $(x_n)_{n \geq 1}$ exists such that $\delta_-(\{k(n) \mid n \in \mathbb{N}\}) = \alpha > 0$, then by Proposition 1.6, we get $i(x_n, L) \geq \alpha$ and therefore

$$\sum_{k=1}^N i(x_n, L_k) + i(x_n, L) \geq \sum_{k=1}^N \alpha_k + \alpha > 1$$

which contradicts Proposition 1.7.

2 Lagrange operators on discontinuous functions

We consider the classical Lagrange operators at the Chebyshev nodes.

The n -th Lagrange operator is defined by

$$L_n f(x) = \sum_{k=1}^n \ell_{n,k}(x) f(x_{n,k}),$$

for every $f : [-1, 1] \rightarrow \mathbb{R}$ and $x \in [-1, 1]$, where for $k = 1, \dots, n$

$$x_{n,k} = \cos \theta_{n,k}, \quad \theta_{n,k} = \frac{(2k-1)\pi}{2n},$$

are the Chebyshev nodes and

$$\ell_{n,k}(x) = \prod_{i \neq k} \frac{x - x_{n,i}}{x_{n,k} - x_{n,i}}$$

are the corresponding fundamental polynomials.

Identifying the variable $x \in [-1, 1]$ with $\cos \theta$, with $\theta \in [0, \pi]$, the polynomials $\ell_{n,k}$ may also be expressed in terms of the variable θ as follows

$$\ell_{n,k}(\cos \theta) = \frac{(-1)^{k-1}}{n} \frac{\cos n\theta}{\cos \theta - \cos \theta_{n,k}} \sin \theta_{n,k}.$$

Our aim is to study the behavior of the sequence of Lagrange operators for a particular class of functions having a finite number of points of discontinuity of the first kind.

We begin to consider the function $h_{x_0,d} : [-1, 1] \rightarrow \mathbb{R}$ defined by

$$h_{x_0,d}(x) := \begin{cases} 0, & x < x_0, \\ d, & x = x_0, \\ 1, & x > x_0, \end{cases} \quad x \in [-1, 1], \quad (2.1)$$

where d is a fixed real number.

Before stating our main result, we need to introduce some zeta functions. Firstly, consider the Hurwitz zeta function

$$\zeta(s, a) := \sum_{n=0}^{+\infty} \frac{1}{(n+a)^s} \quad (2.2)$$

for all $s, a \in \mathbb{C}$ such that $\operatorname{Re}[s] > 1$ and $\operatorname{Re}[a] > 0$. The previous series is absolutely convergent and its sum can be extended to a meromorphic function defined for all $s \neq 1$.

Moreover we need to consider also the Lerch zeta function

$$\Phi(x, s, a) := \sum_{n=0}^{+\infty} \frac{e^{2n\pi i x}}{(n+a)^s}$$

where $x \in \mathbb{R}$, $a \in]0, 1]$, $\operatorname{Re}[s] > 1$ if $x \in \mathbb{Z}$ and $\operatorname{Re}[s] > 0$ otherwise. In the special case $x = \frac{1}{2}$, we obtain the Lerch zeta function

$$J(s, a) := \Phi\left(\frac{1}{2}, s, a\right) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(n+a)^s}$$

which is related to the Hurwitz zeta function by the following relation

$$J(s, a) = \frac{1}{2^{s-1}} \zeta\left(s, \frac{a}{2}\right) - \zeta(s, a)$$

for all $s, a \in \mathbb{C}$ such that $0 < a \leq 1$ and $\operatorname{Re}[s] > 1$.

In order to state our main result, we define the function $g :]0, 1[\mapsto \mathbb{R}$ by setting

$$g(x) := \frac{\sin(\pi x)}{\pi} J(1, x), \quad \text{if } x \in]0, 1[.$$

Theorem 2.1 *Let $x_0 = \cos \theta_0 \in]-1, 1[$ and consider the functions $h := h_{x_0, d}$ defined by (2.1). Then, the sequence of functions $(L_n h)_{n \geq 1}$ converges uniformly to h on every compact subsets of $[-1, 1] \setminus \{x_0\}$.*

As regards the behaviour of the sequence $(L_n h(x_0))_{n \geq 1}$ we have

i) *If $\frac{\theta_0}{\pi} = \frac{p}{q}$ with $p, q \in \mathbb{N}$, $q \neq 0$ and $\operatorname{GCD}(p, q) = 1$, then*

$$i\left(L_n h(x_0); g\left(\frac{2m+1}{2q}\right)\right) = \frac{1}{q}, \quad m = 0, \dots, q-1$$

if q is odd and

$$i(L_n h(x_0); d) = \frac{1}{q}, \quad i\left(L_n h(x_0); g\left(\frac{m}{q}\right)\right) = \frac{1}{q}, \quad m = 1, \dots, q-1$$

if q is even.

ii) if $\frac{\theta_0}{\pi}$ is irrational and if $A \subset \mathbb{R}$ is a Peano-Jordan measurable set, then

$$i(L_n h(x_0); A) = |g^{-1}(A)| ,$$

where $|\cdot|$ denotes the Peano-Jordan measure.

PROOF. Let $a = \cos \theta_1 \in [-1, x_0[$ and $x = \cos \theta \in [-1, a]$; for sufficiently large $n \geq 1$ there exists k_0 such that

$$0 < \theta_{n,k_0} \leq \theta_0 < \theta_{n,k_0+1} < \theta_1 \leq \theta \leq \pi$$

and therefore

$$0 < \cos \theta_0 - \cos \theta_1 \leq \cos \theta_{n,k_0} - \cos \theta .$$

We have $L_n h(\cos \theta) = \sum_{k=1}^{k_0-1} \ell_{n,k}(\cos \theta) + d \ell_{n,k_0}(\cos \theta)$ if $\theta_{n,k_0} = \theta_0$, and $L_n h(\cos \theta) = \sum_{k=1}^{k_0} \ell_{n,k}(\cos \theta)$ if $\theta_{n,k_0} < \theta_0$; hence

$$\begin{aligned} L_n h(\cos \theta) &= \sum_{k=1}^{k_0} \frac{(-1)^{k-1}}{n} \frac{\cos n\theta}{\cos \theta - \cos \theta_{n,k}} \sin \theta_{n,k} + (d-1) \chi_{\{\theta_{n,k_0}\}}(\theta_0) \ell_{n,k_0}(\cos \theta) \\ &= \sum_{k=1}^{k_0} \frac{(-1)^k}{n} \frac{\cos n\theta}{\cos \theta_{n,k} - \cos \theta} \sin \theta_{n,k} + (d-1) \chi_{\{\theta_{n,k_0}\}}(\theta_0) \ell_{n,k_0}(\cos \theta) . \end{aligned}$$

The function $t \rightarrow \frac{\sin t}{\cos t - \cos \theta}$ is positive and monotone increasing on the interval $[0, \theta[$; since $0 < \theta_{n,k} < \theta_{n,k+1} < \theta$ for every $1 \leq k \leq k_0$, we have

$$\begin{aligned} |L_n h(x)| &= |L_n(h)(\cos \theta)| \\ &\leq \left| \frac{\cos n\theta}{n} \frac{\sin \theta_{n,k_0}}{\cos \theta_{n,k_0} - \cos \theta} \right| + |d-1| \left| \frac{\cos n\theta}{n} \frac{\sin \theta_{n,k_0}}{\cos \theta_{n,k_0} - \cos \theta} \right| \\ &\leq \frac{1 + |d-1|}{n} \frac{1}{\cos \theta_0 - \cos \theta_1} . \end{aligned}$$

It follows that $(L_n h)_{n \geq 1}$ converges uniformly to h in $[-1, a]$.

Now let $b = \cos \theta_2 \in]x_0, 1[$ and $x = \cos \theta \in [b, 1]$. For sufficiently large $n \geq 1$ there exists k_0 such that

$$0 \leq \theta \leq \theta_2 < \theta_{n,k_0} \leq \theta_0 < \theta_{n,k_0+1} < 2\pi$$

and consequently

$$0 < \cos \theta_2 - \cos \theta_0 \leq \cos \theta - \cos \theta_{n,k_0+1} .$$

Then

$$\begin{aligned}
|1 - L_n h(x)| &= |1 - L_n h(\cos \theta)| = \left| \sum_{k=1}^n \ell_{n,k}(\cos \theta) - \sum_{k=1}^{k_0} \ell_{n,k}(\cos \theta) h(\cos \theta_{n,k}) \right| \\
&= \left| \sum_{k=k_0+1}^n \frac{(-1)^k}{n} \frac{\cos n\theta}{\cos \theta - \cos \theta_{n,k}} \sin \theta_{n,k} - (d-1) \chi_{\{\theta_{n,k_0}\}}(\theta_0) \ell_{n,k_0}(\cos \theta) \right| \\
&\leq \left| \frac{\cos n\theta}{n} \frac{\sin \theta_{n,k_0+1}}{\cos \theta - \cos \theta_{n,k_0+1}} \right| + |d-1| \left| \frac{\cos n\theta}{n} \frac{\sin \theta_{n,k_0}}{\cos \theta - \cos \theta_0} \right| \\
&\leq \frac{1 + |d-1|}{n} \frac{1}{\cos \theta_2 - \cos \theta_0},
\end{aligned}$$

since the function $t \rightarrow \frac{\sin t}{\cos \theta - \cos t}$ is positive and monotone decreasing in $[\theta, \pi]$ and $\theta < \theta_{n,k-1} < \theta_{n,k} < \pi$ for every $k_0 + 1 \leq k \leq n$. So $(L_n h)_{n \geq 1}$ converges uniformly to h also in $[b, 1]$.

Now, we study the behavior of $(L_n h(x_0))_{n \geq 1}$.

For sufficiently large $n \geq 1$ there exists k_0 such that $\theta_{n,k_0} \leq \theta_0 < \theta_{n,k_0+1}$.

Let us denote $\sigma_n = n \frac{\theta_0 - \theta_{n,k_0}}{\pi}$. From $\frac{2k_0-1}{2n}\pi \leq \theta_0 < \frac{2k_0+1}{2n}\pi$ we have that $0 \leq \sigma_n < 1$; then

$$n = \frac{\pi}{\theta_0} (\sigma_n + k_0 - 1/2)$$

and moreover

$$k_0 \leq n \frac{\theta_0}{\pi} + \frac{1}{2} \leq k_0 + 1,$$

that is $k_0 = \left\lceil n \frac{\theta_0}{\pi} + \frac{1}{2} \right\rceil$ and

$$\sigma_n = n \frac{\theta_0}{\pi} + \frac{1}{2} - \left\lceil n \frac{\theta_0}{\pi} + \frac{1}{2} \right\rceil. \quad (2.3)$$

If x_0 is a Chebyshev node, that is $\theta_0 = \theta_{n,k_0}$ and $\sigma_n = 0$, then

$$L_n h(\cos \theta_0) = d. \quad (2.4)$$

If x_0 is not a Chebyshev node we have $\theta_0 < \theta_{n,k_0}$, $0 < \sigma_n < 1$ and

$$L_n h(\cos \theta_0) = \sum_{k=1}^{k_0} \ell_{n,k}(\cos \theta_0). \quad (2.5)$$

Let us consider the case where x_0 is not a Chebyshev node and observe that

$$\begin{aligned}
\sum_{k=1}^{k_0} \ell_{n,k}(\cos \theta_0) &= \sum_{k=1}^{k_0} \frac{(-1)^{1-k}}{n} \frac{\cos n\theta_0}{\cos \theta_0 - \cos \theta_{n,k}} \sin \theta_{n,k} \\
&= \sum_{k=1}^{k_0} \frac{(-1)^{k_0-k+1}}{n} \frac{\sin(n\theta_0 - k_0\pi + \pi/2)}{\cos \theta_0 - \cos \theta_{n,k}} \sin \theta_{n,k} \\
&= \sum_{k=1}^{k_0} \frac{(-1)^{k_0-k+1}}{n} \frac{\sin(n(\theta_0 - \theta_{n,k_0}))}{\cos \theta_0 - \cos \theta_{n,k}} \sin \theta_{n,k}.
\end{aligned}$$

Setting $m = k_0 - k$ we have

$$\begin{aligned}\theta_0 - \theta_{n,k} &= \theta_0 - \theta_{n,k_0-m} = \theta_0 - \frac{2(k_0 - m) - 1}{2n} \pi = \theta_0 - \theta_{n,k_0} + \frac{m}{n} \pi \\ &= \frac{\pi}{n} (\sigma_n + m)\end{aligned}$$

and consequently

$$\begin{aligned}\sum_{k=1}^{k_0} \ell_{n,k}(\cos \theta_0) &= \sum_{m=0}^{k_0-1} \frac{(-1)^{m+1}}{n} \frac{\sin(\pi \sigma_n)}{\cos \theta_0 - \cos \theta_{n,k_0-m}} \sin \theta_{n,k_0-m} \\ &= \frac{\sin(\pi \sigma_n)}{\pi} \sum_{m=0}^{k_0-1} \frac{(-1)^m}{\sigma_n + m} \\ &\quad + \frac{\sin(\pi \sigma_n)}{n} \sum_{m=0}^{k_0-1} (-1)^{m+1} \left[\frac{\sin \theta_{n,k_0-m}}{\cos \theta_0 - \cos \theta_{n,k_0-m}} + \frac{n}{\pi} \frac{1}{\sigma_n + m} \right] \\ &= \frac{\sin(\pi \sigma_n)}{\pi} \sum_{m=0}^{k_0-1} \frac{(-1)^m}{\sigma_n + m} \\ &\quad + \frac{\sin(\pi \sigma_n)}{n} \sum_{m=0}^{k_0-1} (-1)^{m+1} \left[\frac{\sin \theta_{n,k_0-m}}{\cos \theta_0 - \cos \theta_{n,k_0-m}} + \frac{1}{\theta_0 - \theta_{n,k_0-m}} \right] \\ &= \frac{\sin(\pi \sigma_n)}{\pi} \sum_{m=0}^{k_0-1} \frac{(-1)^m}{\sigma_n + m} \\ &\quad + \frac{\sin(\pi \sigma_n)}{n} \sum_{m=0}^{k_0-1} (-1)^{m+1} g_{\theta_0}(\theta_{n,k_0-m})\end{aligned}\tag{2.6}$$

where $\theta_{n,k_0-m} \in [\theta_{n,1}, \theta_{n,k_0}] \subset]0, \theta_0[$ and the function $g_{\theta_0} :]0, \theta_0[\rightarrow \mathbb{R}$ is defined by setting

$$g_{\theta_0}(x) := \frac{\sin x}{\cos \theta_0 - \cos x} + \frac{1}{\theta_0 - x}, \quad x \in]0, \theta_0[.$$

The function g_{θ} is monotone decreasing and bounded since

$$\lim_{x \rightarrow 0^+} g_{\theta_0}(x) = \frac{1}{\theta_0} < \infty, \quad \lim_{x \rightarrow \theta_0^-} g_{\theta_0}(x) = \frac{1}{2} \cot(\theta_0) < \infty.$$

For all $n \geq 1$ and $\sigma \in [0, 1[$, consider the function $f_n : [0, 1[\rightarrow \mathbb{R}$ defined by setting

$$f_n(\sigma) := \begin{cases} \frac{\sin(\pi \sigma)}{\pi} \sum_{m=0}^{k_0-1} \frac{(-1)^m}{\sigma + m} \\ \quad + \frac{\sin(\pi \sigma)}{n} \sum_{m=0}^{k_0-1} (-1)^{m+1} g_{\theta_0}(\theta_{n,k_0-m}), & \text{if } \sigma \in]0, 1[, \\ d, & \text{if } \sigma = 0; \end{cases}$$

taking into account (2.4), (2.5) and (2.6) we have that $L_n h(\cos \theta_0) = f_n(\sigma_n)$.

For all $\sigma \in]0, 1[$

$$\begin{aligned}
|f_n(\sigma) - g(\sigma)| &\leq \left| \frac{\sin(\pi\sigma)}{\pi} \sum_{m=k_0}^{\infty} \frac{(-1)^m}{\sigma + m} \right| + \frac{\sin(\pi\sigma)}{n} (|g_{\theta_0}(\theta_{n,1})| + |g_{\theta_0}(\theta_{n,k_0})|) \\
&\leq \frac{\sin(\pi\sigma)}{\pi} \left| \frac{(-1)^{k_0}}{\sigma + k_0} \right| + \frac{\sin(\pi\sigma)}{n} (|g_{\theta_0}(\theta_{n,1})| + |g_{\theta_0}(\theta_{n,k_0})|) \\
&\leq \frac{1}{\pi k_0} + \frac{1}{n} (|g_{\theta_0}(\theta_{n,1})| + |g_{\theta_0}(\theta_{n,k_0})|);
\end{aligned}$$

where the righthand side is independent of $\sigma \in]0, 1[$ and it converges to 0 as $n \rightarrow \infty$ since

$$\lim_{n \rightarrow \infty} g_{\theta_0}(\theta_{n,1}) = \lim_{x \rightarrow 0^+} g_{\theta_0}(x) = \frac{1}{\theta_0} < \infty$$

and

$$\lim_{n \rightarrow \infty} g_{\theta_0}(\theta_{n,k_0}) = \lim_{x \rightarrow \theta_0^-} g(x) = \frac{1}{2} \cot(\theta_0) < \infty.$$

Then we can conclude that the sequence $(f_n)_{n \geq 1}$ converges uniformly on $[0, 1[$ to the function $\tilde{g} : [0, 1[\rightarrow \mathbb{R}$ defined as follows

$$\tilde{g}(x) := \begin{cases} g(x), & \text{if } x \in]0, 1[, \\ d, & \text{if } x = 0. \end{cases}$$

Now, we will construct q subsequences $(L_{k_m(n)} h(x_0))_{n \geq 1}$, $m = 0, \dots, q-1$, of $(L_n h(x_0))_{n \geq 1}$ with density $\frac{1}{q}$ such that

$$\lim_{n \rightarrow \infty} L_{k_m(n)} h(x_0) = \tilde{g} \left(\frac{m + q/2 - [q/2]}{q} \right) \text{ for all } m = 0, \dots, q-1.$$

Fix $m = 0, \dots, q-1$; since $GCD(p, q) = 1$ we can set $k_m(n) := l + nq$, where $l \in \{ -[q/2], \dots, [q/2] \}$ is such that $lp \equiv m - [q/2] \pmod{q}$, that is there exists $s \in \mathbb{Z}$ such that $lp = sq + m - [q/2]$.

So, consider $(L_{k_m(n)} h(x_0))_{n \geq 1}$ and observe that for all $m = 0, \dots, q-1$, we have $\delta(\{k_m(n) \mid n \in \mathbb{N}\}) = \frac{1}{q}$. It follows, for all $n \geq 1$

$$\begin{aligned}
\sigma_{k_m(n)} &= (l + nq) \frac{p}{q} + \frac{1}{2} - \left[(l + nq) \frac{p}{q} + \frac{1}{2} \right] \\
&= s + \frac{m + q/2 - [q/2]}{q} + np - \left[s + \frac{m + q/2 - [q/2]}{q} + np \right] \\
&= \frac{m + q/2 - [q/2]}{q}
\end{aligned}$$

since $s, np \in \mathbb{Z}$, while $0 \leq \frac{m+q/2-[q/2]}{q} < 1$ because $0 \leq q/2 - [q/2] < 1$. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} L_{k_m(n)} h(x_0) &= \lim_{n \rightarrow \infty} f_n(\sigma_{k_m(n)}) \\ &= \lim_{n \rightarrow \infty} f_n\left(\frac{m+q/2-[q/2]}{q}\right) = \tilde{g}\left(\frac{m+q/2-[q/2]}{q}\right). \end{aligned}$$

Therefore, by Proposition 1.6, we have that for all $m = 1, \dots, q$

$$i\left(L_n h(x_0), \tilde{g}\left(\frac{m+q/2-[q/2]}{q}\right)\right) \geq \frac{1}{q}.$$

Now, we have q different statistical limits with index $\frac{1}{q}$, so by Proposition 1.7 it necessarily follows

$$i\left(L_n h(x_0); \tilde{g}\left(\frac{m+q/2-[q/2]}{q}\right)\right) = \frac{1}{q}.$$

In particular, if q is even and $m = 0$, for every $n \geq 1$ we have $\theta_0 = \theta_{k_m(n), k_0}$ and

$$\lim_{n \rightarrow \infty} L_{k_m(n)} h(x_0) = \tilde{g}(0) = d.$$

This completes the proof of part i).

Finally we consider the case where $\frac{\theta_0}{\pi}$ is irrational. First, we observe that from (2.3) and Example 1.4 (ii), we have

$$\delta(\{n \in \mathbb{N} \mid \sigma_n \in J\}) = |J|$$

for every Peano-Jordan measurable set $J \subset [0, 1[$.

Let $A \subset \mathbb{R}$ be a bounded Peano-Jordan measurable set. Since $(f_n)_{n \geq 1}$ converges uniformly to g in $]0, 1[$, for every $\varepsilon > 0$ there exists $\nu_\varepsilon \geq 1$ such that $f_n(\sigma) \in A + B_\varepsilon$ whenever $n \geq \nu_\varepsilon$ and $\sigma \in g^{-1}(A)$. So

$$\{n \geq \nu_\varepsilon \mid \sigma_n \in g^{-1}(A)\} \subset \{n \geq \nu_\varepsilon \mid L_n h(x_0) \in A + B_\varepsilon\}$$

and we can conclude that

$$|g^{-1}(A)| \leq \delta_-(\{n \in \mathbb{N} \mid L_n h(x_0) \in A + B_\varepsilon\}).$$

Hence

$$|g^{-1}(A)| \leq i(L_n h(x_0); A). \quad (2.7)$$

In order to show the converse inequality, we argue by contradiction and assume that $|g^{-1}(A)| < i(L_n h(x_0); A)$; then we can find $\delta > 0$ such that

$$|g^{-1}(A + B_\delta)| = i(L_n h(x_0); A).$$

The map $\delta \mapsto |A + B_\delta|$ is monotone increasing and continuous for $\delta \geq 0$.

By Proposition 1.7, since $\overline{A} \cap \overline{(A + B_{\delta/2})^c} = \emptyset$, we have

$$i(L_n h(x_0); A) + i(L_n h(x_0); (A + B_{\delta/2})^c) \leq 1.$$

Using (2.7), we get $|g^{-1}((A + B_{\delta/2})^c)| \leq i(L_n h(x_0); (A + B_{\delta/2})^c)$; then

$$|g^{-1}(A + B_\delta)| + |g^{-1}((A + B_{\delta/2})^c)| \leq 1$$

and consequently, taking into account that $g^{-1}(\mathbb{R}) =]0, 1[$,

$$|g^{-1}(A + B_\delta)| \leq 1 - |g^{-1}((A + B_{\delta/2})^c)| = |g^{-1}(A + B_{\delta/2})|$$

which yields a contradiction, since the map $\delta \mapsto |A + B_\delta|$ is monotone increasing.

□

At this point, using Theorem 2.1, we are able to study the behavior of Lagrange operators on larger classes of functions, namely on the space $BV([-1, 1])$ of functions of bounded variation having a finite number of points of discontinuity and on the space $C_\omega + H$ where C_ω denotes the space of all functions $f \in C([-1, 1])$ satisfying the Dini-Lipschitz condition $\omega(f, \delta) = o(|\log \delta|^{-1})$, and H is the linear space generated by

$$\{h_{x_0, d} \mid x_0 \in]-1, 1[, d \in \mathbb{R}\}.$$

Observe that if $f \in C_\omega + H$ there exists at most a finite number of points x_1, \dots, x_N of discontinuity with finite left and right limits $f(x_i - 0)$ and $f(x_i + 0)$, $i = 1, \dots, N$.

Then we can state the following theorem.

Theorem 2.2 *Let $f \in BV([-1, 1])$, or alternatively $f \in C_\omega + H$, with a finite number N of points of discontinuity of the first kind at $x_1, \dots, x_N \in]-1, 1[$. For every $i = 1, \dots, N$ consider $\theta_i \in]0, \pi[$ such that $x_i = \cos \theta_i$, $d_i := f(x_i)$ and define the function*

$$g_i(x) := f(x_i - 0) + (f(x_i + 0) - f(x_i - 0))g(x).$$

Then, the sequence $(L_n f)_{n \geq 1}$ converges uniformly to f on every compact subset of $] -1, 1[\setminus \{x_1, \dots, x_N\}$.

Moreover for all $i = 1, \dots, N$ the sequence $(L_n f(x_i))_{n \geq 1}$ has the following behavior

i) if $\frac{\theta_i}{\pi} = \frac{p}{q}$ with $p, q \in \mathbb{N}$, $q \neq 0$ and $\text{GCD}(p, q) = 1$, then

$$i \left(L_n f(x_i); g_i \left(\frac{2m+1}{2q} \right) \right) = \frac{1}{q}, \quad m = 0, \dots, q-1;$$

if q is odd and

$$i(L_n f(x_i); d_i) = \frac{1}{q},$$

$$i \left(L_n f(x_i); g_i \left(\frac{m}{q} \right) \right) = \frac{1}{q}, \quad m = 1, \dots, q-1.$$

if q is even.

ii) If $\frac{\theta_i}{\pi}$ is irrational and if $A \subset \mathbb{R}$ is a Peano-Jordan measurable set, then

$$i(L_n f(x_i); A) = |g_i^{-1}(A)|.$$

PROOF. We assume $x_1 < \dots < x_N$. We can write $f = F + \sum_{k=1}^N c_k h_k$, where $F \in BV([-1, 1]) \cap C([-1, 1])$ or, alternatively, $F \in C_\omega$ and $h_i := h_{x_i, \tilde{d}_i}$ for every $i = 1, \dots, N$.

Since F is continuous we have

$$f(x_i + 0) - \sum_{k=1}^{i-1} c_k - c_i = F(x_i + 0) = F(x_i - 0) = f(x_i - 0) - \sum_{k=1}^{i-1} c_k,$$

from which

$$c_i = f(x_i + 0) - f(x_i - 0)$$

and

$$F(x_i) = f(x_i - 0) - \sum_{k=1}^{i-1} c_k. \quad (2.8)$$

Moreover

$$\begin{aligned} d_i &= f(x_i) = F(x_i) + \sum_{k=1}^{i-1} c_k h_k(x_i) + c_i \tilde{d}_i \\ &= F(x_i) + \sum_{k=1}^{i-1} c_k + (f(x_i + 0) - f(x_i - 0)) \tilde{d}_i \\ &= f(x_i - 0) + (f(x_i + 0) - f(x_i - 0)) \tilde{d}_i. \end{aligned}$$

and hence

$$\tilde{d}_i = \frac{d_i - f(x_i - 0)}{f(x_i + 0) - f(x_i - 0)}.$$

The first part of our statement is a trivial consequence of the linearity of Lagrange interpolation operators. Indeed $L_n F \rightarrow F$ uniformly on compact subsets of $] -1, 1[$ by [10, Theorem 3.1, p. 24] (see also [8]) if $F \in BV([-1, 1]) \cap C([-1, 1])$ and by [9, Theorem 14.4, p. 335] in the case $F \in C_\omega$.

Moreover for every $k = 1, \dots, N$, by Theorem 2.1 $L_n h_k \rightarrow h_k$ converges uniformly to h_k on compact subsets of $[-1, 1] \setminus \{x_k\}$. Then $L_n f = L_n F + \sum_{k=1}^N c_k L_n h_k$ converges uniformly to f on compact subsets of $] -1, 1[\setminus \{x_1, \dots, x_N\}$.

Now we establish property i). We fix a point x_i of discontinuity and following the same line of the proof of Theorem 2.1 we construct the subsequences $(k_m(n))_{n \geq 1}$, $m = 1, \dots, q$. Since

$$L_{k_m(n)} f(x_i) = L_{k_m(n)} F(x_i) + \sum_{\substack{k=1 \\ k \neq i}}^N c_k L_{k_m(n)} h_k(x_i) + c_i L_{k_m(n)} h_i(x_i)$$

and taking into account (2.8) and that $F \in BV([-1, 1]) \cap C([-1, 1])$ (or alternatively $F \in C_\omega$), from Theorem 2.1 the right-hand side converges to

$$\begin{aligned} F(x_i) + \sum_{k=1}^{i-1} c_k h_k(x_i) + c_i g\left(\frac{2m+1}{2q}\right) \\ = f(x_i - 0) + (f(x_i + 0) - f(x_i - 0)) g\left(\frac{2m+1}{2q}\right) \\ = g_i\left(\frac{2m+1}{2q}\right) \end{aligned}$$

for $m = 0, \dots, q-1$, if q is odd.

Analogously, if q is even, $(L_n f(x_i))_{n \geq 1}$ converges to

$$f(x_i - 0) + (f(x_i + 0) - f(x_i - 0)) \tilde{d}_i = d_i$$

with index $\frac{1}{q}$ and to

$$f(x_i - 0) + (f(x_i + 0) - f(x_i - 0)) g\left(\frac{m}{q}\right) = g_i\left(\frac{m}{q}\right)$$

with index $\frac{1}{q}$ for $m = 1, \dots, q-1$.

Finally, we prove property ii). For every $i = 1, \dots, N$ we have

$$L_n f(x_i) = L_n F(x_i) + \sum_{\substack{k=1 \\ k \neq i}}^N c_k L_n h_k(x_i) + c_i L_n h_i(x_i) .$$

For the sake of simplicity let us denote

$$y_n := L_n f(x_i) , \quad z_n := L_n F(x_i) + \sum_{\substack{k=1 \\ k \neq i}}^N c_k L_n h_k(x_i) , \quad x_n := c_i L_n h_i(x_i) ;$$

thus $y_n = z_n + x_n$ and, from (2.8),

$$z := F(x_i) + \sum_{k=1}^{i-1} c_k h_k(x_i) = f(x_i - 0) .$$

Now, we can apply [10, Theorem 3.1, p. 24] if $F \in BV([-1, 1]) \cap C([-1, 1])$ and [9, Theorem 14.4, p. 335] if $F \in C_\omega$ and in any case, from Theorem 2.1, we get $z_n \rightarrow z$ and $i(c_i^{-1} x_n; A) = |g^{-1}(A)|$ for every bounded Peano-Jordan measurable set $A \subset \mathbb{R}$. Hence $i(x_n; A) = |g^{-1}(c_i^{-1} A)|$, that is

$$|g^{-1}(c_i^{-1} A)| = \inf_{\varepsilon > 0} \delta_-(\{n \in \mathbb{N} \mid x_n \in A + B_\varepsilon\}) .$$

Fix $\varepsilon > 0$; if $x_n \in A + B_\varepsilon$, from the equality $x_n = y_n - z_n$ we get

$$y_n \in A + B_\varepsilon + z_n = A + B_\varepsilon + z + z_n - z .$$

Now, let $\nu \in \mathbb{N}$ such that $|z_n - z| < \varepsilon$ for all $n \geq \nu$, Then for every $n \geq \nu$ we have $z_n - z \in B_\varepsilon$ and consequently $y_n \in A + B_{2\varepsilon} + z$. Then

$$\{n \geq \nu \mid x_n \in A + B_\varepsilon\} \subset \{n \geq \nu \mid y_n \in A + B_{2\varepsilon} + z\},$$

that is

$$\delta_-(\{n \in \mathbb{N} \mid x_n \in A + B_\varepsilon\}) \leq \delta_-(\{n \in \mathbb{N} \mid y_n \in A + B_{2\varepsilon} + z\}). \quad (2.9)$$

On the other hand, if $y_n \in A + B_{2\varepsilon} + z$, then $x_n = y_n - z_n \in A + B_{2\varepsilon} + z - z_n$. In this case for every $n \geq \nu$, we have $z - z_n \in B_\varepsilon$ and therefore $x_n \in A + B_{3\varepsilon}$; hence

$$\delta_-(\{n \in \mathbb{N} \mid x_n \in A + B_{3\varepsilon}\}) \geq \delta_-(\{n \in \mathbb{N} \mid y_n \in A + B_{2\varepsilon} + z\}). \quad (2.10)$$

Taking the infimum over $\varepsilon > 0$ in (2.9) and (2.10) we can conclude that $i(x_n, A) \leq i(y_n, A + z) \leq i(x_n, A)$ which yields

$$i(y_n, A + z) = i(x_n, A) = |g^{-1}(c_i^{-1}A)|.$$

We conclude that $i(y_n, A) = |g^{-1}(c_i^{-1}(A - z))| = \left| g^{-1} \left(\frac{A - f(x_i - 0)}{f(x_i + 0) - f(x_i - 0)} \right) \right| = |g_i^{-1}(A)|$ for every bounded Peano-Jordan measurable set $A \subset \mathbb{R}$. \square

3 Shepard operators on discontinuous functions

Let $s \geq 1$; the n -th Shepard operator $S_{n,s}$ is defined by

$$S_{n,s}f(x) = \frac{\sum_{k=0}^n f\left(\frac{k}{n}\right) \left|x - \frac{k}{n}\right|^{-s}}{\sum_{k=0}^n \left|x - \frac{k}{n}\right|^{-s}}$$

for every $f : [0, 1] \rightarrow \mathbb{R}$ and $x \in [0, 1]$.

For the general properties of Shepard operators we refer to [3]. In particular we point out that the sequence $(S_{n,s}f)_{n \geq 1}$ converges uniformly to f for every $f \in C([-1, 1])$ (see [3, Theorem 2.1]).

Our aim is to study the behavior of the sequence of Shepard operators for bounded functions which have a finite number of points of discontinuity of the first kind and are continuous elsewhere.

Also in this case we begin by considering the function $h_{x_0,d} : [0, 1] \rightarrow \mathbb{R}$ defined by

$$h_{x_0,d}(x) := \begin{cases} 1, & x < x_0, \\ d, & x = x_0, \\ 0, & x > x_0, \end{cases} \quad x \in [0, 1], \quad (3.1)$$

where $x_0 \in [0, 1]$ and $d \in \mathbb{R}$ are fixed.

In order to state the convergence properties of the sequence $(S_{n,s}h_{x_0,d})_{n \geq 1}$, for every $s > 1$ we consider the function $g_s : [0, 1] \mapsto \mathbb{R}$ defined by setting, for every $x \in [0, 1]$,

$$g_s(x) = \frac{\zeta(s, x)}{\zeta(s, x) + \zeta(s, 1 - x)},$$

where ζ is the Hurwitz zeta function defined by (2.2).

We have the following result.

Theorem 3.1 *Let $x_0 \in [0, 1]$ and $h := h_{x_0,d}$ be defined by (3.1). Then for every $s \geq 1$ the sequence $(S_{n,s}h)_{n \geq 1}$ converges uniformly to h on every compact subset of $[0, 1] \setminus \{x_0\}$.*

As regards the behavior of the sequence $(S_{n,s}h(x_0))_{n \geq 1}$ we have

i) *If $x_0 = \frac{p}{q}$ with $p, q \in \mathbb{N}$, $q \neq 0$ and $\text{GCD}(p, q) = 1$, then*

$$i(S_{n,s}h(x_0); d) = \frac{1}{q}$$

and further

$$\begin{aligned} s > 1 &\implies i\left(S_{n,s}h(x_0); g_s\left(\frac{m}{q}\right)\right) = \frac{1}{q}, \quad m = 1, \dots, q-1, \\ s = 1 &\implies i\left(S_{n,s}h(x_0); \frac{1}{2}\right) = 1 - \frac{1}{q}. \end{aligned}$$

ii) *if x_0 is irrational and if $A \subset \mathbb{R}$ is a Peano-Jordan measurable set, then*

$$\begin{aligned} s > 1 &\implies i(S_{n,s}h(x_0); A) = |g_s^{-1}(A)|, \\ s = 1 &\implies i\left(S_{n,s}h(x_0); \frac{1}{2}\right) = 1. \end{aligned}$$

Moreover, in the case $s = 1$, there exist subsequences of $(S_{n,s}h(x_0))_{n \geq 1}$ converging to 0 and 1 (consequently the set of indices of these subsequences must have density zero).

PROOF. We set $k_0 = [nx_0]$, so that, for sufficiently large $n \geq 1$, $\frac{k_0}{n} \leq x_0 < \frac{k_0+1}{n}$.

Let $a \in [0, x_0[$ and $x \in [0, a]$, we have

$$S_{n,s}h(x) = \frac{\sum_{k=0}^{k_0} \left|x - \frac{k}{n}\right|^{-s} + \chi_{\{x_0n\}}(k_0)(d-1)|x - x_0|^{-s}}{\sum_{k=0}^n \left|x - \frac{k}{n}\right|^{-s}},$$

then

$$\begin{aligned} &S_{n,s}h(x) - 1 \\ &= \frac{\sum_{k=0}^{k_0} \left|x - \frac{k}{n}\right|^{-s} + \chi_{\{x_0n\}}(k_0)(d-1)|x - x_0|^{-s} - \sum_{k=0}^n \left|x - \frac{k}{n}\right|^{-s}}{\sum_{k=0}^n \left|x - \frac{k}{n}\right|^{-s}} \\ &= \frac{\sum_{k=k_0+1}^n \left|x - \frac{k}{n}\right|^{-s} + \chi_{\{x_0n\}}(k_0)(d-1)|x - x_0|^{-s}}{\sum_{k=0}^n \left|x - \frac{k}{n}\right|^{-s}}. \end{aligned} \tag{3.2}$$

If $k > k_0$, since $x \leq a < x_0 < \frac{k}{n}$, we have that $\frac{k}{n} - x > x_0 - a > 0$, moreover $x_0 - x > x_0 - a$, then

$$\begin{aligned} \sum_{k=k_0+1}^n \left| x - \frac{k}{n} \right|^{-s} + \chi_{\{x_0\}}(k_0) |d-1| |x - x_0|^{-s} \\ \leq (n - k_0) |x_0 - a|^{-s} + \chi_{\{x_0\}}(k_0) |d-1| |x_0 - a|^{-s} \\ \leq (n - [nx_0] + |d-1|) |x_0 - a|^{-s} < +\infty. \end{aligned} \quad (3.3)$$

On the other hand we have

$$\begin{aligned} \sum_{k=0}^n \left| x - \frac{k}{n} \right|^{-s} &\geq \sum_{[nx] < k \leq n} \left| x - \frac{k}{n} \right|^{-s} = n^s \sum_{[nx] < k \leq n} (k - nx)^{-s} \\ &\geq n^s \sum_{[nx] < k \leq n} (k - [nx])^{-s} = n^s \sum_{m=1}^{n-[nx]} m^{-s} \geq n^s \sum_{m=1}^{n-[na]} m^{-s}. \end{aligned} \quad (3.4)$$

From (3.3) and (3.4), we can rewrite (3.2) as follows

$$|S_{n,s}h(x) - 1| \leq \frac{(n - [nx_0] + |d-1|) |x_0 - a|^{-s}}{n^s \sum_{m=1}^{n-[na]} m^{-s}}$$

where the righthand side converges to 0 as $n \rightarrow \infty$. Indeed if $s > 1$,

$$\lim_{n \rightarrow \infty} \frac{(n - [nx_0] + d)}{n^s} = 0,$$

while if $s = 1$

$$\lim_{n \rightarrow \infty} \sum_{m=1}^{n-[na]} m^{-1} = +\infty.$$

Now let $x_0 < a \leq x \leq 1$; using the same arguments we get

$$|S_{n,s}h(x)| \leq \frac{([nx_0] + 1 + |d-1|) |x_0 - a|^{-s}}{n^s \sum_{m=1}^{[na]+1} m^{-s}}.$$

Then the sequence of functions $(S_{n,s}h)_{n \geq 1}$ converges uniformly to h on every compact subset of $[0, 1] \setminus \{x_0\}$.

Now we focus our attention on the behavior of the sequence $(S_{n,s}h(x_0))_{n \geq 1}$. Let us denote $\sigma_n = nx_0 - k_0$, that is $\sigma_n = nx_0 - [nx_0]$ and observe that $0 \leq \sigma_n < 1$. If x_0 coincides with a node then

$$S_{n,s}h(x_0) = d$$

otherwise

$$\begin{aligned}
S_{n,s}h(x_0) &= \frac{\sum_{k=0}^{k_0} |x_0 - \frac{k}{n}|^{-s}}{\sum_{k=0}^n |x_0 - \frac{k}{n}|^{-s}} \\
&= \frac{\sum_{k=0}^{k_0} |nx_0 - k|^{-s}}{\sum_{k=0}^n |nx_0 - k|^{-s}} = \frac{\sum_{k=0}^{k_0} |nx_0 - k_0 + k_0 - k|^{-s}}{\sum_{k=0}^n |nx_0 - k_0 + k_0 - k|^{-s}} \\
&= \frac{\sum_{m=0}^{k_0} (\sigma_n + m)^{-s}}{\sum_{m=0}^{k_0} (\sigma_n + m)^{-s} + \sum_{m=0}^{n-k_0-1} (1 - \sigma_n + m)^{-s}} .
\end{aligned}$$

Now, consider the function $f_{n,s} : [0, 1[\rightarrow \mathbb{R}$ defined by setting

$$f_{n,s}(\sigma) := \begin{cases} \frac{\sum_{m=0}^{k_0} (\sigma + m)^{-s}}{\sum_{m=0}^{k_0} (\sigma + m)^{-s} + \sum_{m=0}^{n-k_0-1} (1 - \sigma + m)^{-s}} , & \text{if } \sigma \in]0, 1[, \\ d , & \text{if } \sigma = 0 . \end{cases}$$

We have $S_{n,s}h(x_0) = f_{n,s}(\sigma_n)$.

If $s > 1$, the sequence $(f_{n,s})_{n \geq 1}$ converges uniformly to the function g_s^* given by

$$g_s^*(\sigma) = \begin{cases} d , & \text{if } \sigma = 0 , \\ g_s(\sigma) , & \text{if } \sigma \in]0, 1[. \end{cases}$$

If $s = 1$ the sequence $(f_{n,s})_{n \geq 1}$ converges pointwise to

$$g_1^*(\sigma) = \begin{cases} d , & \text{if } \sigma = 0 , \\ \frac{1}{2} , & \text{if } \sigma \in]0, 1[\end{cases}$$

and the convergence is uniform on every compact subset of $]0, 1[$.

Arguing similarly to the proof of Theorem 2.1 we obtain property *i*) for $s \geq 1$ and *ii*) for $s > 1$.

As regards the case $s = 1$, we consider an interval $[a, b] \subset]0, 1[$. Since the sequence $(f_{n,1})_{n \geq 1}$ converges uniformly in $[a, b]$, for every $\varepsilon > 0$ there exists $\nu \in \mathbb{N}$ such that $|f_{n,1}(x) - \frac{1}{2}| \leq \varepsilon$ whenever $n \geq \nu$ and $x \in [a, b]$. Then

$$\{n \in \mathbb{N} \mid n \geq \nu, \sigma_n \in [a, b]\} \subset \left\{ n \in \mathbb{N} \mid S_{n,1}h(x_0) \in \left] \frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon \right[\right\} ,$$

and since the sequence σ_n is equidistributed on $]0, 1[$

$$\delta_- \left\{ n \in \mathbb{N} \mid S_{n,1}h(x_0) \in \left] \frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon \right[\right\} \geq b - a$$

that is $i(S_{n,1}h(x_0); \frac{1}{2}) \geq b - a$ for every $0 < a < b < 1$.

It follows $i(S_{n,1}h(x_0); \frac{1}{2}) = 1$.

Moreover from [1, Theorem 2] we have that

$$\limsup_{n \rightarrow \infty} S_{n,1}h(x_0) = \max \left\{ \lim_{x \rightarrow x_0^-} h(x), \lim_{x \rightarrow x_0^+} h(x) \right\} = 1$$

and

$$\liminf_{n \rightarrow \infty} S_{n,1}h(x_0) = \min \left\{ \lim_{x \rightarrow x_0^-} h(x), \lim_{x \rightarrow x_0^+} h(x) \right\} = 0 ,$$

then we can construct subsequences of $(S_{n,1}h(x_0))_{n \geq 1}$ converging to 0 and 1, but, thanks to Remark 1.8 and Proposition 1.6, they must have the set of indices with density zero. \square

Finally, we extend Theorem 3.1 to a larger class of functions.

Theorem 3.2 *Let f be a bounded function with N points of discontinuity of the first kind at $x_1, \dots, x_N \in]0, 1[$ and continuous elsewhere. For every $i = 1, \dots, N$ consider $d_i := f(x_i)$ and define the function*

$$g_{s,i}(x) := f(x_i + 0) + (f(x_i - 0) - f(x_i + 0))g_s(x) .$$

Then, for every $s \geq 1$ the sequence $(S_{n,s}f)_{n \geq 1}$ converges uniformly to f on every compact subset of $[0, 1] \setminus \{x_1, \dots, x_N\}$.

Moreover for all $i = 1, \dots, N$ the sequence $(S_{n,s}f(x_i))_{n \geq 1}$ has the following behavior

i) *If $x_i = \frac{p}{q}$ with $p, q \in \mathbb{N}$, $q \neq 0$ and $\text{GCD}(p, q) = 1$, then*

$$i(S_{n,s}h(x_i); d_i) = \frac{1}{q}$$

and further

$$\begin{aligned} s > 1 &\implies i\left(S_{n,s}f(x_i); g_{s,i}\left(\frac{m}{q}\right)\right) = \frac{1}{q}, \quad m = 1, \dots, q-1, \\ s = 1 &\implies i\left(S_{n,s}f(x_i); \frac{f(x_i + 0) + f(x_i - 0)}{2}\right) = 1 - \frac{1}{q}. \end{aligned}$$

ii) *if x_i is irrational and if $A \subset \mathbb{R}$ is a Peano-Jordan measurable set, then*

$$\begin{aligned} s > 1 &\implies i(S_{n,s}f(x_i); A) = |g_{s,i}^{-1}(A)|, \\ s = 1 &\implies i\left(S_{n,s}f(x_i); \frac{f(x_i + 0) + f(x_i - 0)}{2}\right) = 1. \end{aligned}$$

Moreover, in the case $s = 1$, there exist subsequences of $(S_{n,s}f(x_i))_{n \geq 1}$ converging to $f(x_i - 0)$ and $f(x_i + 0)$ whose set of indices has density zero.

PROOF. We assume $x_1 < \dots < x_N$. For every $k = 1, \dots, N$, we set $c_k := f(x_k - 0) - f(x_k + 0)$ and $\tilde{d}_k := \frac{d_k - f(x_k + 0)}{f(x_k - 0) - f(x_k + 0)}$; consequently we can write $f = F + \sum_{k=1}^N c_k h_k$, where $F \in C([0, 1])$ and $h_k := h_{x_k, \tilde{d}_k}$ for every $k = 1, \dots, N$. So $f \in C([0, 1]) + H$ and since Shepard operators converge uniformly in $C([0, 1])$ (see e.g. [3, Theorem 2.1]), we can argue as in Theorem 2.2 using Theorem 3.1 in place of Theorem 2.1. \square

References

- [1] Bojanic, R. , Della Vecchia, B. and Mastroianni, G., *On the approximation of bounded functions with discontinuities of the first kind by generalized Shepard operators*. Acta Math. Hungarica **85** (1-2) (1999), 29–57.
- [2] Connor, J. S., *The statistical and strong p -Cesàro convergence of sequences*. Analysis **8** (1988), 47–63.
- [3] Della Vecchia, B. and Mastroianni, G., *On functions approximation by Shepard-type operators – a survey*, in Approximation Theory, Wavelets and Applications (Maratea, 1994), 335–346. NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **454**, Kluwer Acad. Publ., Dordrecht, 1995.
- [4] Duman, O. and Orhan, C., *Statistical approximation by positive linear operators*. Studia Math. **161** (2) (2004), 187–197.
- [5] Fridy, J. A., *On statistical convergence*. Analysis **5** (1985), 301–313.
- [6] Fridy, J. A., *Statistical limit points*. Proc. Amer. Math. Soc. **118** (1993), 1187–1192.
- [7] Fridy, J. A. and Orhan, C., *Statistical limit superior and limit inferior*. Proc. Amer. Math. Soc. **125** (1997), 3625–3631.
- [8] Mastroianni, G., Milovanović, G., Interpolation processes - Basic theory and applications. *Springer Monographs in Mathematics*, Springer-Verlag Berlin Heidelberg, 2008.
- [9] Szegő, G., Orthogonal Polynomials. *Colloquium Publ., vol. XXIII*, Amer. Math. Soc., Providence, RI, 1959; Russian translation: Fizmatlit, Moscow, 1962.
- [10] Vertesi, P., *Lagrange interpolation for continuous functions of bounded variation*. Acta Math. Acad. Scient. Hung. **35** (1-2) (1980), 23–31.
- [11] Williams, K. S. and Nan-Yue, Z., *Special values of the Lerch zeta function and the evaluation of certain integrals*. Proc. Amer. Math. Soc. **119** (1) (1993), 35–49.
- [12] Whittaker, E. T. and Watson, G. N., A course of modern analysis, 4th ed. *Cambridge Univ. Press*, Cambridge and New York, (1963).